# A new approach to teach mathematics for engineers (1). Numerical sets 

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#### Abstract

This paper is the first in a series in which the authors propose to change the face of teaching mathematics in a technical university, especially for first year students who are preparing to become civil engineers. The paper refers to the teaching of concepts related to some sets of real numbers. The main idea is that the exposure rigorous, technique, specific to teach mathematics in a college math is completed by an informal exposure. Thus the authors add historical information, find motivations of the subject discussed, present some applications, mainly in engineering, and states open problems of the field. So, exposure becomes more accessible. Another idea is that even the exposure technique is not usual, the authors preferring gradual exposure, instead of traditional. Also the authors present some computer programs related to the subject and an application in engineering.


Key words: real number, irrational number, transcendental number, perfect number, countable set, uncontable set, bounded set, unbounded set, sampling, signal processing.

## 1. Introduction

The authors of this paper are mathematicians and work in several areas of mathematical research. But at the same time three of them teach mathematics in a technical university. That is why they are interested in finding new approaches to teaching mathematics.

The main idea is that the exposure rigorous, technique, specific to teach mathematics in a college math is completed by an informal exposure. Thus the authors add historical information, find motivations of the subject discussed, present some applications in engineering, and states open problems of the field. So exposure
becomes more accessible. And, in his turn, each student can choose part what attracts him, according to his interest in knowledge and his academic ambition.

Another idea is that even exposure technique is not the usual, the authors preferring gradual exposure, instead of the traditional.

What is the gradual method?
This means that the exposure is constructed in several steps:
1). Definitions and examples - containing all fundamental concepts representing topic dictionary and examples intended to facilitate understanding of these concepts;
2). Classical results - that is some known results meant to establish fundamental properties of concepts of dictionary, the connections between these concepts and some algorithms;
3). Proofs - in the role of theoretical exercises, proofs of some of these results in particular those that manipulate notions of subject dictionary;
4). Examples and solved exercises - examples of applications of these results and solved exercises, exploring traditional types of applications;
5). Questionnaire - as a recap and to fix the main notions and the fundamental algorithms follows a series of questions to which answering students can compose their own summary of the course;
6). Proposed exercises - a large number of proposed exercises, accompanied by indications and answers.

What is the advantage of gradual method?
Each step traveled by the student to initiate, for example, in Calculus, is an important step in understanding and deepening the course. Thus, for each topic discussed, students may be limited to knowledge dictionary, or add, successively, knowledge of examples and results, or can develop computing techniques, and solving adequate exercises, modeled on those already solved.

If the subject allows, some applications in engineering or/and in economics are given. What to say especially for those who are not mathematicians is that some topics not directly applicable to engineering can not be omitted from the curriculum, because they are necessary for mathematical construction.

Next we present briefly how we can teach in a gradual manner several topics related to sets of real numbers, an issue which is at the foundation of the Calculus.

## 2. Description of a classic course taught in a gradual manner

### 2.1. Definitions and examples

In this paper all sets will be nonempty. In the following, "D." stands for "Dictionary".
D. Usual numerical sets: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} ; \mathbb{R}=\mathbb{Q} \cup(\mathbb{R} \backslash \mathbb{Q})$.
D. Countable set: a set A which can be put in one-to-one correspondence with $\mathbb{N}$, so what is such that there exists a bijective function $f: \mathbb{N} \rightarrow A$. By denoting $f(n)=a_{n}(n \in \mathbb{N})$, it follows that $A=\left(a_{n}\right)_{n \in \mathbb{N}}$, that is the elements of the set $A$ can be "listed".

## Examples of countable sets <br> $\mathbb{N}, \mathbb{Z}, \mathbb{Z}_{-}, \mathbb{Q}_{+}, \mathbb{Q}_{-}$, <br> $\mathbb{Q}, 2 \mathbb{Z}(=\{2 k \mid k \in \mathbb{Z}\})$


D. Uncountable set: Any set that is not a countable set.

| Examples of uncountable sets |
| :---: |
| $\mathbb{R}, \mathbb{R}_{+}, \mathbb{R}_{-}, \mathbb{R} \backslash \mathbb{Q},(a, b),[a, b)$, |
| $(a, b],[a, b]$ (with $a, b \in \mathbb{R}, a \leq b)$ |

D. Bounded above set (respectively bounded below set): a set $A \subset \mathbb{R}$ such that there exists $b \in \mathbb{R}$ with $x \leq b$ ( $x \geq b$, respectively) $\forall x \in A$. The element $b$ is called an upper bound (respectively a lower bound) for $A$.

D. Bounded set: a set $A \subset \mathbb{R}$, that is both a bounded above set and a bounded below set, that is $\exists a, b \in \mathbb{R}$, with $a \leq x \leq b, \forall x \in A \Leftrightarrow \exists M \in \mathbb{R}_{+}$, with $|x| \leq M$, $\forall x \in A$.


A set which is not a bounded set is called an unbounded set.

| Examples of unbounded sets: |
| :---: |
| The intervals |
| $(-\infty, a),(-\infty, a],[a,+\infty)$, with |
| $a \in \mathbb{R} ;$ the interval $(-\infty,+\infty)$. |


| Examples of bounded sets: |
| :---: |
| The intervals |
| $(a ; b),[a ; b),(a ; b]$ and $[a ; b]$, with |
| $a \leq b, a, b \in \mathbb{R}$. |

D. The supremum of a bounded above set $A \subset \mathbb{R}$ : the real number, denoted by $\sup (A)$ which is the least upper bound of the set $A$, that is the least real number that is greater than or equal to all elements of $A$. (We can show that $\sup (A)$ exists and it is unique, for any nonempty and bounded above set $A \subset \mathbb{R}$.)
D. The infimum of a bounded below set $A \subset \mathbb{R}$ : the real number, denoted by $\inf (A)$ which is the greatest lower bound of the set $A$, that is the greatest real number that is less than or equal to all elements of $A$. (We can show that $\inf (A)$ exists and it is unique, for any nonempty and bounded below set $A \subset \mathbb{R}$.)

### 2.2. Axioms

## Axiom of Archimedes

Classic enunciation: (i) $\forall x, y \in \mathbb{R}, y>0 \Rightarrow \exists n \in \mathbb{N}^{*}$ such that $n y>x$.

"Here, the minimum value for $n$ is 5 "
Equivalent enunciation: (ii) $\forall z \in \mathbb{R}, \exists$ !(that is there exists a unique) $m \in \mathbb{R}$ so that $m \leq z<m+1$. Note $m=[z]$ (read the entire part of $z$ ).


Axiom of Cantor
Classic enunciation: (i) $\forall\left(a_{n}\right)_{n \in \mathbb{N}} \subset \mathbb{Q}$ and $\left(b_{n}\right)_{n \in \mathbb{N}} \subset \mathbb{Q}$ such that:
$a_{0} \leq a_{1} \leq \ldots \leq a_{n} \leq \ldots \leq b_{n} \leq \ldots \leq b_{1} \leq b_{0}$, and
$\exists \lim _{n}\left(b_{n}-a_{n}\right)$ and $\lim _{n}\left(b_{n}-a_{n}\right)=0$
$\Rightarrow \exists!c \in \mathbb{R}$ so that $a_{n} \leq c \leq b_{n}, \forall n \in \mathbb{N}$.

Equivalent enunciation: (ii) Any decreasing sequence of closed intervals $\left(I_{n}\right)_{n \geq 1}$ (here "decreasing" meaning that $I_{n+1} \subseteq I_{n}, \forall n \geq 1$ ), with the extremities of intervals in $\mathbb{Q}$ and having the sequence of lengths converging to 0 , has the intersection $\bigcap_{n>1} I_{n}$ reduced to a point.


$$
\bigcap_{n \geq 1} I_{n}=\{c\}
$$

### 2.3. Classical results: Propositions ("P."), Theorems ("T."), Corollaries ("C.")

P1. (Density of $\mathbb{Q}$ in $\mathbb{R}$ ): $\forall x, y \in \mathbb{R}$, with $x<y \Rightarrow \exists r \in \mathbb{Q}$, such that $x<r<y$ (that is between any two real numbers there exists a least one rational number).


C2. $\forall x, y \in \mathbb{R}$, with $x<y \Rightarrow$ there are an infinity of rational numbers $\left(r_{n}\right)_{n \geq 1}$ so that: $x<r_{n}<y, \forall n \in \mathbb{N}$.


P3. A finite union of countable sets is a countable set. (More general, even a countable union of countable sets is countable.)

P4. $\mathbb{Q}$ is a countable set.

Note that in 4) of section "6." we will highlight this result which can be solved by the so called "Cantor's enumeration of a countable collection of countable sets", and we will give a computer program, too.

P5. The interval $[0 ; 1]=\{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$ is an uncountable set.
C6. The interval $[a ; b]$ (with $a<b$ in $\mathbb{R}$ ) is an uncountable set.
C7. (Density of $\mathbb{R} \backslash \mathbb{Q}$ in $\mathbb{R}) \forall x, y \in \mathbb{R}$, with $x<y \Rightarrow \exists s \in \mathbb{R} \backslash \mathbb{Q}$, so that $x<s<y$ (that is between any two real numbers there is at least one irrational number).


T8. (Nested intervals Theorem):
$\forall\left(a_{n}\right)_{n \geq 1} \subset \mathbb{R},\left(b_{n}\right)_{n \geq 1} \subset \mathbb{R}$ such that:

1) $\left.a_{1} \leq a_{2} \leq \ldots \leq a_{n} \leq \ldots \leq b_{n} \leq \ldots \leq b_{2} \leq b_{1}\right\} \Rightarrow \exists!c \in \mathbb{R}$ such that
2) $\exists \lim _{n}\left(b_{n}-a_{n}\right)$ and $\lim _{n}\left(b_{n}-a_{n}\right)=0 \quad a_{n} \leq c \leq b_{n} \forall n \geq 1$

T9. Any nonempty bounded above set (bounded below set, respectively) $A \subset \mathbb{R}$, has a supremum (respectively an infimum).

P10. If $A \subset \mathbb{R}$ and $M \in \mathbb{R}$, then:
$M=\sup (A) \Leftrightarrow \forall x \in A, x \leq M$, and $\forall \varepsilon>0, \exists x_{\varepsilon} \in A$ such that $M-\varepsilon<x_{\varepsilon}$.

$m=\inf (A) \Leftrightarrow \forall x \in A, x \geq m$, and $\forall \varepsilon>0, \exists x_{\varepsilon}^{\prime} \in A$ such that $m+\varepsilon<x_{\varepsilon}^{\prime}$.


For the proofs of above results, see for example [2]. (In this book there are also many examples, solved and proposed problems, which manipulate the notions introduced in the dictionary of the subject and the related results.)

To summarize the lesson content and indicate what are the most important items a minimal number of questions are formulated in the following

### 2.4. Questionnaire (Numerical sets)

1). What are the usual numerical sets? How can they be described?
2). What were the goals of the following extensions: $\mathbb{N} \subset \mathbb{Z}$ (that is, from set of natural numbers to the set of integers), $\mathbb{Z} \subset \mathbb{Q}, \mathbb{Q} \subset \mathbb{R}$ ?
3). How can define the (usual) order relation " $\leq$ " on the set $\mathbb{R}$ ? (Note: " $\leq "$ must have the following properties: (R) reflexivity, (AS) antisymmetry, (T) transitivity, where:
(R) $x \leq x, \forall x \in \mathbb{R}$; (AS) $x \leq y$ and $y \leq x \Rightarrow x=y$; (T) $x \leq y$ and $y \leq z \Rightarrow x \leq z$ ).
4). What is a totally ordered commutative algebraic field? (recall what means algebraic "field" and note that "total ordered" means that any two elements $x, y$ are comparable in this field endowed with " $\leq$ ", that is we have or $x \leq y$, or $y \leq x$ ).
5). Stated in words the previously axioms and results from the section 2.3.
6). What are the properties of the sequences $\left(a_{n}\right)_{n}$ and $\left(b_{n}\right)_{n}$ from Cantor's axiom?
7). State the Cantor's axiom by using the notion of intervals.
8). Show that $\mathbb{N}$ and $\mathbb{Z}$ are countable, but $\mathbb{R}$ is not countable.

## 3. Motivation

### 3.1. Motivation to introduce the usual numerical sets

It is well known that the usual numerical sets (namely the sets of natural numbers, and respectively of integers, of rational numbers, of real numbers and of complex numbers) were gradually extended, starting with the set of natural numbers:
$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$.
So, starting from the set $\mathbb{N}$ of all natural numbers (whose introduction has resulted in the counting) reached the set $\mathbb{Z}$ of integer numbers, due to the impossibility to resolve in $\mathbb{N}$ some algebraic equations with the coefficients in $\mathbb{N}$. For example, the equation $x+1=0(\Leftrightarrow x=-1)$ has no solution in $\mathbb{N}$.

Now, starting from the set $\mathbb{Z}$ reached the set $\mathbb{Q}$ of rational numbers, due to the impossibility to solve in $\mathbb{Z}$ some algebraic equations with coefficients in $\mathbb{Z}$. For example, the equation $2 x+1=0\left(\Leftrightarrow x=-\frac{1}{2}\right)$ has no solution in $\mathbb{Z}$.

Then, starting from the set $\mathbb{Q}$ reached the set $\mathbb{R}$ of real numbers, due to the impossibility to solve in $\mathbb{Q}$ some algebraic equations with coefficients in $\mathbb{Q}$. For example, the equation $x^{2}-2=0(\Leftrightarrow x= \pm \sqrt{2})$ has no solution in $\mathbb{Q}$.
(The notion of real number is the central notion in mathematics. Can we get real numbers, namely positive numbers, and if we measure the length of segments arranged on an axis.)

Finally, starting from the set $\mathbb{R}$ reached the set $\mathbb{C}$ of complex numbers, due to the impossibility to solve in $\mathbb{R}$ some algebraic equations with coefficients in $\mathbb{R}$. For example, the equation $x^{2}+1=0(\Leftrightarrow x= \pm i$, where $i=\sqrt{-1}$ is the imaginary unit $)$ has no solution in $\mathbb{R}$.

Remarks. In the decimal floating-point representation, positive rational numbers have the decimals (the fractional part) in one of the following cases: 1). all decimals are equal to zero (that is the decimal fraction represents an integer); 2). there is only a finite number of nonzero decimals, that is the rational number has a finite decimal expansion; 3). the fractional part is infinite but a part of the decimal expansion is repeating, that is the rational number has a reccuring decimal expansion.

A rational number in the situation ,3)" is called a reccuring decimal or a repeating decimal and can be so that:

3a) it becomes periodic just after the decimal point
(examples: $0,121212 \ldots=0,(12)=\frac{12}{99}$, and $0,325325325 \ldots=0,(325)=\frac{325}{999}$ );
3b) its decimal representation becomes periodic not immediately after the decimal point
(examples: $0,1(45)=0,1454545 \ldots=\frac{145-1}{990}$, and
$\left.0,42(0923)=0,4209230923 \ldots=\frac{420923-42}{999900}\right)$.
The main extension in the „chain" (1) of sets is from $\mathbb{Q}$ to $\mathbb{R}$, the numbers added to the rational numbers, being the irrational numbers, that is all numbers in $\mathbb{R} \backslash \mathbb{Q}$.

Remark that the irrational numbers are all real numbers that can not be written as ordinal fractions, that is all real numbers that are not as $\frac{a}{b}$ with $a, b \in \mathbb{Z}, b \neq 0$.

In decimal representation, the irrational numbers have an infinite decimal expansion, but there is no part that repeats. Perhaps, the most known irrational numbers are $\pi$ (= the rate between the length of a cercle and its diameter.), $e$, and $\sqrt{2}$.

### 3.2. Motivation for introducing the concepts of equipotent sets, countable sets and uncountable sets

We know that the set theory plays a central role in mathematics. When we talk about finite sets, things are very clear. Thus for a finite set $A$, the number of its elements (or its cardinal number, denoted by card $A$ ) is finite. But if we are talking about infinite sets, things become less accessible because our finite minds put limits in our thinking. For example, it is little difficult to understand that the (infinite) set of natural numbers can be put in one-to-one correspondence (that is element by element) with an own subset that does not contain all natural numbers, for example with the set $2 \mathbb{N}=\{2 k \mid k \in \mathbb{N}\}$ of even numbers. (Note that $2 \mathbb{N} \subset \mathbb{N}$, hence $2 \mathbb{N} \neq \mathbb{N}$.)

This has been observed for the first time, it seems, by Galileo Galilei. Indeed we can define the function

$$
f: \mathbb{N} \rightarrow 2 \mathbb{N} \text { by } f(n)=2 n, \text { for any } n \in \mathbb{N} ;
$$

of course, this function is bijective, that is injective $(f(n)=f(m) \Rightarrow n=m)$ and surjective $(f(\mathbb{N})=2 \mathbb{N}$, where $f(\mathbb{N})=\{f(n) \mid n \in \mathbb{N}\})$. For finite sets, this is not possible, that there is no finite set $A$ to be put in one-to-one correspondence with its own subset, $B$, different than it (that is $B \subset A$, hence $B \neq A$ ).
(Galileo Galilei, 1564-1642, was an Italian physicist, mathematician, astronomer, and philosopher who played a major role in the Scientific Revolution that is the emergence of modern science during the early modern period, when developments in mathematics, physics, astronomy, biology, medicine, and chemistry transformed views of society and nature. His achievements include improvements to the telescope and consequent astronomical observations. Galileo has been called the "Father of Modern Observational Astronomy", the "Father of Modern Physics", the "Father of Science", and "the Father of Modern Science".)

Georg Cantor was the one who introduced the notion of equipotent sets, as being two sets $C$ and $D$ such that there is a bijective function $f: C \rightarrow D$ (which is denoted by $C \sim D$ ). So we remark that $\mathbb{N} \sim 2 \mathbb{N}$, although apparently $2 \mathbb{N}$ has fewer elements than $\mathbb{N}$ (because, obviously, $2 \mathbb{N} \subset \mathbb{N}$ - indeed, for example, $1 \in \mathbb{N}$ but $1 \notin 2 \mathbb{N}$ ), that is these two sets ( $\mathbb{N}$ and $2 \mathbb{N}$ ) can be put in one-to-one correspondence.
(Georg Ferdinand Ludwig Philipp Cantor, 1845-1918, was a German mathematician, best known as the inventor of set theory, which has become a fundamental theory in mathematics; Cantor established the importance of one-to-one correspondence between the members of two sets, defined infinite and proved that the real numbers are "more numerous" than the natural numbers)

Already know from previous ones, that the sets which can be put in an one-toone correspondence with $\mathbb{N}$ ( for example, $2 \mathbb{N}$ ) are called countable sets. It is natural to ask whether "All infinite sets of real numbers are countable?". The answer is "NO!". Hence there are infinite sets which are not countable. For example any interval $((a, b),[a, b),(a, b],[a, b])$ of real numbers, and even the set R of all real numbers.

Another example illustrating the limits of our finite thinking is related to what is called " the density of $\mathbb{Q}$ (set of rational numbers) in $\mathbb{R}$ " (see P1. in Introduction). This means that for any real number, there are rational numbers, no matter how close to him. In consequence, representing numbers of $\mathbb{R}$ and those of $\mathbb{Q}$ by points on an axis, we see no difference: "holes" absence due to irrational numbers (that is numbers of $\mathbb{R} \backslash \mathbb{Q}$ ), like $\sqrt{2}$, do not notice because rational numbers closer together.

### 3.3. Motivation for the notions of maximum, minimum, supremum and infimum

Firstly, we add to the definitions of supremum and infimum (see 2.1.), two other definitions. We recall that any bounded above (bounded below, respectively) subset of the set of all real numbers has supremum (respectively, infimum).
D. If $A$ is a bounded above set, containing its supremum (sup (A)), then this supremum is the maximum of $A$, denoted by max (A).
D. If $A$ is a bounded below set, containing its infimum (inf $(A)$ ), then this infimum is the minimum of $A$, denoted by $\min (A)$.

The ideea of motivation that follows was suggested in [5].
Now we ask what is the maximum of the function $f: D \rightarrow \mathbb{R}$, defined by the
following formula: $f(x)=\left\{\begin{array}{l}\frac{x^{2}+x}{x},-2 \leq x<0 \\ \frac{-x^{2}+x}{x}, 0<x \leq 2\end{array}\right.$, where $D=[-2,2] \backslash\{0\}$
or, equivalent, what is the maximum of the set $A=\{f(x) \mid x \in D\}$, of all values of function $f$, that is the highest value of the set $A$ ( if this value exists, in $\mathbb{R}!$ ).

Observe that :

$$
f(x)=\left\{\begin{array}{l}
x+1,-2 \leq x<0 \\
-x+1,0<x \leq 2
\end{array}\right.
$$



Graphical representation of the function $f$.

The graph is composed of segments [AB) and [CB). Observing the graph of above function is obvious that $f(x) \leq 1$ (is true even the strict inequality, that is $f(x)<1$ ), for any $x \in D$. In addition, 1 is the smallest number greater than all values $f(x)$ (belonging to the set $A$ ).

We could even say that 1 is "the maxim value" of function $f$. Note however that 1 is not value of this function (because there is no number $x$ in the domain of $f$, with $f(x)=1)$.

In this case, use the word "supremum" instead of "maximum". Denote the maximum of the set $A=\{f(x) \mid x \in[-2,0) \cup(0,2]\}$ by $\max (A)=M$ and the supremum of this set by $\sup (A)=s$.

Similarly we can speak about the "minimum" and the "infimum" of the set $A$ (if these elements exist in $\mathbb{R}$ ), noting them by $\min (A)=m$, and $\inf (A)=i$, respectively.

It follows: $M$ does not exist, but $s=1, m=-1$ and $i=-1$. (The element " $s$ " is not reached on the domain of $f$.)

We observe also that $i \leq s$ (actually $i<s$ ). Moreover this inequality is because the set $A$ is nonempty.

Remark. Unlike the situation when the set $A \subset \mathbb{R}$ not admits the supremum and/or infimum_in $\mathbb{R}=(-\infty,+\infty)$, every nonempty set $A \subseteq \mathbb{R}$ has supremum, $s$, and infimum, $i$, in $\mathbb{R}=[-\infty,+\infty]$, with $s \leq+\infty$ and $i \geq-\infty$; hence, possibly, $s=+\infty$ (if the set $A$ is not bounded above in $\mathbb{R}$ ) and/or $i=-\infty$ (if the set $A$ is not bounded below in $\mathbb{R}$ ). It shows that the set $A$ has supremum and infimum in $\mathbb{R}=(-\infty,+\infty)$, if and only if it is (nonempty and) bounded, that is bounded above and bounded below.

## 4. Historical Notes

Note that we have found the knowledge for this section, consulting [1], [3], [4], [6], [7], [8], [1'], [7'] and [8'].

Firstly we remember something about real numbers, complex numbers, algebraic numbers and transcendental numbers.

A real number is a value that represents a quantity along a continuous line. The real numbers include all the rational numbers, such as the integer -7 and the fraction $\frac{9}{5}$, and all the irrational numbers (such as $\sqrt{2}=1.41421356 \ldots$, an irrational algebraic number, but also, for example, $\pi=3.14159265 .$. , a transcendental number). Real numbers can be thought of as points on an infinitely long line called the number line or real line, where the points corresponding to integers are equally spaced. Any real number can be determined by a possibly infinite decimal representation such as that of 9.03675, where each consecutive digit is measured in units one tenth the size of the previous one. The real line can be thought of as a part of the complex plane and correspondingly, complex numbers include real numbers as a special case.

These descriptions of the real numbers are not sufficiently rigorous by the modern standards of pure mathematics. The discovery of a suitably rigorous definition of the real numbers, including the realization that a better definition was needed, was one of the most important developments of 19th century mathematics.

A complex number is a number that can be put in the form $a+b i$, where $a$ and $b$ are real numbers and $i$ is called the imaginary unit $\left(i^{2}=-1\right)$. In this expression, $a$ is called the real part and $b$ the imaginary part of the complex number. Complex numbers extend the idea of the one-dimensional number line to the two-dimensional complex plane by using the horizontal axis for the real part and the vertical axis for the imaginary part. The complex number $a+b i$ can be identified with the point $(a, b)$ in the complex plane. A complex number whose real part is zero is said to be purely
imaginary, whereas a complex number whose imaginary part is zero is a real number. In this way the complex numbers contain the ordinary real numbers while extending them in order to solve problems that cannot be solved with real numbers alone.

An algebraic number is a number that is a root of a non-zero polynomial in one variable with rational coefficients (or equivalently, by bringing rational coefficients to the same denominator and then eliminating it, with integer coefficients). Numbers (such as $\pi$ ) that are not algebraic are said to be transcendental; "almost all" real and complex numbers are transcendental. (Here "almost all" has the sense "all but except a countable set".)

A transcendental number is a (possibly complex) number that is not algebraic. The most prominent examples of transcendental numbers are $\pi$ and $e$. Though only a few classes of transcendental numbers are known (in part because it can be extremely difficult to show that a given number is transcendental), transcendental numbers are not rare. Indeed, "almost all" real and complex numbers are transcendental, since the set of algebraic numbers is countable while the sets of real and complex numbers are both uncountable. All real transcendental numbers are irrational, since all rational numbers are algebraic. The converse is not true: not all irrational numbers are transcendental; for example the square root of 2 is irrational but not a transcendental number, since it is a solution of the polynomial equation $x^{2}-2=0$.

In the sequel of this section we mention something about history of real numbers.

### 4.1. In Antiquity and pre-Antiquity

## Early use of rational numbers in ancient Egypt

Ordinary fractions, that is fractions whose numerators and denominators are integers (for example $\frac{1}{2},-\frac{3}{4},-\frac{9}{2}$ ) were used by the Egyptians around the year 1000 BC.

## Consideration of the concept of irrational numbers in ancient India

During the ancient scriptures of Hindu the geometric manuscript entitled "Rules of chords" ("Sulba Sutras", around the year 600 BC ) includes what may be the first use of irrational numbers (numbers that cannot be written in the form of ordinary fractions). The concept of irrationality was implicitly accepted by Indian mathematicians in the early period, starting with Manava (author of the manuscript "Sulba Sutras", 750-690 BC). It is believed that these mathematicians realized that the square root of certain numbers such as 2 and 61 can not be exactly determined. However, Carl B. Boyer (1968-see [1]) believed that such statements are unlikely to be true and even that they are not true.

## Consideration of the concept of irrational numbers in ancient Greece

Around the year 500 BC, Greek mathematicians led by Pytagoras of Samos (Greek mathematician and philosopher of ionic origin, $570-495 \mathrm{BC}$, founder of the religious movement, esoteric and metaphysical, known as Pytagoreanism, cult which was considerably influenced by mathematics) realized the need to introduce irrational numbers in mathematics, such as $\sqrt{2}$.

In ancient Greece occurred the first demonstration of the existence of irrational numbers as background reports of two incommensurable segments (two segments which "have" no common extent). This demonstration, erroneously attributed to Pythagoras, is due to Hippasus of Metapontum (fifth century BC), pythagorean philosopher who lived about a century after Pythagoras.

At that time Hippasus'ideas were not very well received because they denied the hypothesis that rational numbers and geometry are inseparable, which was later held by Euclid (also known as Euclid of Alexandria, Greek mathematician, often regarded as the "Father of Geometry", lived around 300 BC ). We mention that the most important treatise written by Euclid, called Euclid's Elements, consisting of 13 books and containing definitions, postulates (axioms), propositions (theorems and constructions) and demonstrations of sentences, is one of the most influential works in the history of mathematics and it has served as the main textbook to teach mathematics (especially geometry), from the time of publication until the end of the nineteenth century and even early twentieth century.

The discovery of the existence of incommensurable segments was related to the passage in time from discreet to continuous. This is another problem in ancient Greek mathematics. Zeno of Elea (Greek philosopher, lived in southern Italy, 490-430 BC) was the first to identify contradictions in mathematical thinking time and has formulated four famous paradoxes.

Another step towards awareness of the concept of irrational number was made by Eudoxus of Cnidus (Greek astronomer and mathematician, 410 or 408-355 or 347 BC ), who considered both reports of commensurable and incommensurable segments. He was the first to consider the notion of proportion.

### 4.2. In the Middle Ages

The number zero, the negative integers, the rational and the irrational numbers at the Indians, the Chinese and the Arabs

Middle Ages (5th-15th centuries) brought the acceptance of the existence of the number zero, negative numbers and the concepts of integer and rational number (also called fractional number), first by Indian and the Chinese mathematicians (in China, mathematics was established as an independent science from the eleventh century).

In addition to these, the Arab mathematicians who lived in Islamic civilization, roughly between the years 662 and 1600, first treated rational numbers as algebraic
objects, which was made possible by the development of algebra, as a branch of mathematics that uses letters and symbols to represent variables and relations between them (for example equations or inequalities checked by them).

Arabic mathematicians merged the concepts of "number" and "magnitude" (the size of a mathematical object) in a more general concept, that of real numbers. Abū Kamil Shuja ibn Aslam ibn Muhammad Ibn Shujã (latinized as the Auoquamel, Egyptian Muslim mathematician, c.850-930) was the first to consider irrational numbers as solutions of equations of the second degree or the values of the coefficients of such equations, called radicals of the second, third or fourth ordinal form.

Muslim mathematicians have also extended the concept of rational number, criticizing the ideas of Euclid on reports that compare the sizes of the same type only (for example lengths with lengths instead of lengths with surfaces).

In his comments on book 10 of "Elements" of Euclid, in the second half of the ninth century Persian mathematician Al-Māhāni examined and classified quadratic and cubic irrational numbers. (To specify what it means, for example, quadratic irrational numbers, we mention that they are the solutions of second degree algebraic equations with rational coefficients, that are like as $\frac{a+b \sqrt{c}}{d}$, where $a, b, c$ and $d \neq 0$ are integers).

In the 9th century, the Iraqi mathematician and astronomer, Ali ibn Sulayman al-Hashimi (850-900) gave algebraic proofs for the existence of irrational numbers, replacing the geometric justifications before. Many of these concepts and many more related to rational and irrational numbers were accepted by mathematicians in Europe during the twelfth century, when the "great translation" of books written by Arab mathematicians occured. Such translations and takeovers also took place later, for example, in the thirteenth century, when Leonardo Fibonacci took over the horizontal bar in the fractional notation used in the twelfth century in the Islamic inheritance jurisprudence. (Leonardo Pisano Bigollo, probably 1170-1250, also known as Leonardo of Pisa, Leonardo Pisano, Leonardo Bonacci, Leonardo Fibonacci, or, most commonly, simply Fibonacci, was an Italian mathematician, considered by some "the most talented western European mathematician who lived in the Middle Ages. ").

In 14th, 15th and 16th centuries, the Kerala school of astronomy and mathematics in India, discovered infinite series whose number amounts are some irrational numbers (as $\pi$ ) and some irrational values of trigonometric functions.

### 4.3. In modern times (the sixteenth, the seventeenth and eighteenth centuries)

## Modern decimal notation

In the sixteenth century, Simon Stevin (Flemish mathematician and military engineer, 1548-1620), who acted in many scientific and engineering domains both theoretical and practical, created the basis for the modern decimal notation, insisting
that from this point of view there is no difference between rational and irrational numbers.

## The terminology of "real numbers"

In the 17th century, René Descartes (latinized Renatus Cartesius, philosopher, mathematician and French writer, called the Father of Modern Philosophy, 15961650) introduced the terminology of "real numbers" to describe the roots of a polynomial, which are different from other numbers that are "imaginary".

### 4.4. In the eighteenth and nineteenth century

## Irrational numbers. Transcendental numbers

In the eighteenth century and nineteenth century there was much work on irrational numbers and transcendental numbers (the latter being those numbers that are not algebraic, that is not roots of a polynomial, non-identically zero, with coefficients in $\mathbb{Q}$, the set of rational numbers, or equivalent - Why?- with coefficients in the set $\mathbb{Z}$ of integers). Johann Heinrich Lambert (Swiss mathematician, physicist, philosopher and astronomer, 1728-1777), was the first who in 1761 gave a demonstration (unfortunately wrong) that the number $\pi$ can not be rationally (and that $e^{n}$ and $\pi^{n}$ can not be rational, if $n \neq 0$ is a rational number). Adrien-Marie Legendre (French mathematician with numerous contributions in mathematics, 1752-1833) completed this demonstration in 1794 and showed that there is no rational (positive) number $r$ with $\pi=\sqrt{r}$. The first number to be proven transcendental without having been specifically constructed for the purpose was e, by Charles Hermite in 1873. (Charles Hermite, a French mathematician who did research on number theory, quadratic forms, invariant theory, orthogonal polynomials, elliptic functions, and algebra, 1822 -1901).

The name "transcendental" comes from Leibniz, Gottfried Wilhelm von Leibniz (a German mathematician and philosopher, 1646 -1716), who in his work of 1682 proved that $\sin x$ is not an algebraic function of $x$.
(An algebraic function is a function that can be expressed using a finite number of terms, involving only the algebraic operations, namely addition, subtraction, multiplication, division, and raising to a fractional power; an algebraic function in one variable $x$ is a function $y=f(x)$ that satisfies a polynomial equation $a_{n}(x) y^{n}+a_{n-1}(x) y^{n-1}+\ldots+a_{0}(x)=0$ where the coefficients $a_{i}(x)$ are polynomial functions of $x$. A function that is not algebraic is called a transcendental function).

Euler (a Swiss pioneering mathematician and physicist, 1707-1783) was probably the first person who defined transcendental numbers in the modern sense.

Joseph Liouville (a French mathematician, 1809-1882) first proved the existence of transcendental numbers in 1844, and in 1851 gave the first decimal examples such as the Liouville constant

$$
\sum_{k=1}^{\infty} 10^{-k!}=0.1100010000000000000000010000 \ldots
$$

in which the $n$th digit after the decimal point is 1 if $n$ is equal to $k$ ! ( $k$ factorial) for some $k$ and 0 otherwise. Liouville showed that this number is what we now call a Liouville number; this essentially means that it can be more closely approximated by rational numbers than can any irrational algebraic number. Liouville showed that all Liouville numbers are transcendental.

In 1874, Georg Ferdinand Ludwig Philipp Cantor (a German mathematician, 1845-1918) proved that the set of algebraic numbers is countable and the real numbers is uncountable. He also gave a new method for constructing transcendental numbers. In 1878, Cantor published a construction that proves there are as many transcendental numbers as there are real numbers. Cantor's work established the ubiquity of transcendental numbers.

In 1882, Carl Louis Ferdinand von Lindemann (a German mathematician, 1852-1939), published a proof that the number $\pi$ is transcendental. He first showed that $e$ to any nonzero algebraic power is transcendental, and since $e^{i \pi}=-1$ is algebraic (see Euler's identity), $i \pi$ and therefore $\pi$ must be transcendental. This approach was generalized by Karl Theodor Wilhelm Weierstrass (German mathematician, who is often cited as the "Father of Modern Analysis", 1815 -1897). The transcendence of $\pi$ allowed the proof of the impossibility of several ancient geometric constructions involving compass and straightedge, including the most famous one, squaring the circle; this is a problem proposed by ancient geometers. (It is the challenge of constructing a square with the same area as a given circle by using only a finite number of steps with compass and straightedge.)

In 1900, David Hilbert, (a German mathematician, recognized as one of the most influential and universal mathematicians of the 19th and early 20th centuries 1862-1943) posed an influential question about transcendental numbers, Hilbert's seventh problem: If $a$ is an algebraic number, that is not zero or one, and $b$ is an irrational algebraic number, is $a^{b}$ necessarily transcendental? The affirmative answer was provided in 1934 by the Gelfond-Schneider theorem (see section 5.2. below). This work was extended by Alan Baker (born in 1939, an English mathematician) in the 1960s in his work on lower bounds for linear forms in any number of logarithms (of algebraic numbers).

## Abel's impossibility theorem

Paolo Ruffini (Italian mathematician and philosopher, 1765-1822), in 1799 and Niels Henrik Abel (Norwegian mathematician, 1802-1829), in 1842 gave both, AbelRuffini theorem demonstrations, also known as the Abel's impossibility theorem. This result shows that a polynomial equation of degree greater than or equal to 5 can not be
solved by a general formula which uses only arithmetic operations and root extractions in terms of the coefficients. This unlike the equations of degree 2,3,4; for example, for the quadratic equation $a x^{2}+b x+c=0$, with $a \neq 0$, the roots are $x=\frac{-b \pm \sqrt{b^{2}-4 a c}}{2 a}$.

### 4.5. In the nineteenth century and twentieth century

Galois theory, a new domain of algebra
Évariste Galois (French mathematician, 1811-1932) developed techniques that can determine when a given equation could be solved by radicals. At only 21 years old, in 1832, he found a criterion (that is a necessary and sufficient condition) for this problem, opening a new domain of algebra, called later Galois theory, which provides a connection between field theory and group theory; certain problems in field theory can be reduced to group theory, which is in some sense simpler and better understood.

## There exists transcendental numbers

In 1840, Joseph Liouville (French mathematician, 1809-1882) showed that the numbers $e$ and $e^{2}$ can not be roots of a quadratic equation with coefficients in $\mathbb{Z}$.

He then established the existence of transcendental numbers, the proof was later partially modified (in 1873) by Georg Philipp Ludwig Ferdinand Cantor (German mathematician, considered the creator of the modern theory of sets, 1845-1918).

In 1873, Charles Hermite (French mathematician, 1822-1901) first proved that the number $e$ is transcendental, and Ferdinand von Lindemann (German mathematician, 1852-1939) showed in 1882, that $\pi$ is a transcendental number, too.

The demonstration was much simplified by Wilhelm Karl Theodor Weierstrass (German mathematician, 1815-1897, considered the Father of Mathematical Analysis) and still further simplified in 1893 by David Hilbert (German mathematician, recognized as one of most influential and universal mathematicians of the nineteenth century and early twentieth century, 1862-1943).

Finally, the same demonstration (the fact that $\pi$ is transcendental number) became elementary, due to the contribution of Adolf Hurwitz (German mathematician, 1837-1912).

Real numbers set - uncountable. Algebraic numbers set - countable
In the eighteenth century, the development of differential calculus used the whole set of real numbers without this set clearly defined. The first rigorous definition was given in 1871 by Georg Ferdinand Ludwig Philipp Cantor (1845-1918, German mathematician, best known for having invented set theory, which has become a fundamental theory in mathematics).

In 1874 , he showed that the set $\mathbb{R}$ of real numbers is infinite and uncountable, (that is it can not be put in bijective correspondence with the set of natural numbers $\mathbb{N}$ or, equivalently, its elements can not be enumerated). Also, Cantor showed that the set of algebraic numbers (numbers that are roots of a non-identically zero polynomial ecuation, with rational coefficients) is infinite and countable (that is it can be put in bijective correspondence with $\mathbb{N}$ ).

## 5. Open problems concerning the real numbers

Motivation of considering the issue "Open problems concerning the real numbers" and some conjectures comes from the fact that solving these problems may take hundreds of years and even millenia, leading mathematicians go further and leading to the development of mathematics.

In this section, the references are [3], [5], [6], [7], [8], [3'], [7'] and [8'].

### 5.1. Open problems concerning perfect numbers

The oldest (apparently) famous unsolved problems come from number theory and refer to perfect numbers (J.R. Goldman, 1998-see [5]).

A natural number $n$ is called a perfect number if the sum $\sigma(n)$ of all its divisors except $n$ is equal to $n$, that is $\sigma(n)=n$.

Examples of perfect numbers:
$6,28,496,8128,33550336,8589869056,137438691328$.
(Indeed, for example, $6=2 \cdot 3$ and all its divisors except 6 are 1, 2 and 3, their sum being even $6: 1+2+3=6$ ).

The numbers listed above are the first 7 perfect numbers. These numbers grow fast enough. Ranked number 12 has 77 digits!

It is noted that the first 12 perfect numbers are all even numbers. The question naturally is: "Are there any odd perfect numbers?"

Question has not yet received a response and is apparently the oldest known unsolved problem in mathematics.

Another unsolved problem requires an answer to another question: "There are an infinite number of perfect numbers?".

Now we refer to:

## A short history of perfect numbers

It seems that it is not known exactly who first studied perfect numbers, but it certainly happened in ancient times once the idea of "number" aroused curiosity. Thus, perfect numbers were known by Pythagoras and his followers. However the name "perfect" due to these mathematicians and was chosen because of the mystical properties of such a number. So the name perfect numbers has religious or astrological origins. Thus, the first two perfect numbers are 6 and 28, and the earth was "created"
in six days and Moon needs 28 days to surround the Earth. Perfect numbers have been studied by the ancient Hebrew. Rabbi Joseph ben Jekuda Ankin in the twelfth century, recommends the study of perfect numbers in his book "Healing Souls"

Greek mathematician Euclid of Alexandria (300-275 BC) noted that if the number $2 \mathrm{p}-1$ is prime, then $2^{\mathrm{p}-1}\left(2^{p}-1\right)$ is a perfect number. It took 2000 years to show that all even perfect numbers are of this form. Around the year 100 (AD), Nicomachus of Gerasa (60-120, an important mathematician in the ancient world), in his book "Introduction to Arithmetic", gave a classification of numbers based on the concept of perfect number. He also pointed to the first 4 perfect numbers. Nicomachus introduced also superabundant numbers, or numbers deficits that are the numbers $n$ with $\sigma(n)>n$, respectively $\sigma(n)<n$.

Greek philosopher Theon of Smyrna (70-135) around the year 130, also classified natural numbers in: perfect, superabundant and deficits.

In the second millennium, many mathematicians have studied perfect numbers. Among them:

1) René Descartes (1596-1660, latinized, Renatus Cartesius, French philosopher, mathematician and writer; he is considered the Father of Modern Philosophy and Analytic Geometry, which is a mathematical discipline that studies the geometry with algebra methods);
2) Leonhard Euler (1707-1783, Swiss mathematician and physicist, considered to be the dominant force of mathematics of the eighteenth century and one of the most distinguished mathematicians and scientists of all time);
3) James Joseph Sylvester (1814-1897, English mathematician with special contributions in matrix theory, number theory and combinatorics, regarded as the leader of American mathematics in the later half of the nineteenth century).

Note that in "1)" of section "6." we will give a script file for find perfect numbers.

### 5.2. Open problems concerning some irrationals numbers

Firstly we ask the following question: Are there irrational numbers $a, b \in \mathbb{R} \backslash \mathbb{Q}$ such that $a^{b}$ is a rational number?

The answer of this question is „Yes!". To justify this take $a=b=\sqrt{2}$. If $a^{b}=(\sqrt{2})^{\sqrt{2}}$ would be rational, then the above answer is justified. If $(\sqrt{2})^{\sqrt{2}}$ would be irrational, then taking $a=(\sqrt{2})^{\sqrt{2}}$ and $b=\sqrt{2}$ it follows that $a^{b}=\left((\sqrt{2})^{\sqrt{2}}\right)^{\sqrt{2}}=\sqrt{2}^{2}=2$.

Remark. Although the solution above can not decide between the two cases, Gelfond-Schneider's theorem shows that $(\sqrt{2})^{\sqrt{2}}$ is transcendental, hence irrational.

The statement of Gelfond-Schneider theorem is the following: If $a$ and $b$ are algebraic numbers, $a \neq 0, a \neq 1$ and $b$ is not $a$ rational number then $a^{b}$ is $a$ transcendental number.

It is known that the real numbers: $3 \pi+2, \pi+\sqrt{2}, e \sqrt{3}$ are irrational numbers even transcedental numbers, also it is known that the number $e^{\pi}$ is irrational.

The following problems are open problems.

1) It is not known if the numbers $\pi+e$ and $\pi-e$ are or not irrationale numbers. More generally, there is no pair of nonzero integers ( $m, n$ ) which to know whether $m \pi+n e$ is or not irrational.
2) It is also not known whether or not $\pi e, \frac{\pi}{e}, 2^{e}, \pi^{e}, \pi^{\sqrt{2}}, \pi^{\pi}, e^{\pi^{2}}, 2^{e}, e^{e}, \ln \pi$ and the number $\gamma=$ constanta Euler-Mascheroni are or not irrationale numbers. Euler's constant (also called the Euler-Mascheroni's constant) is a mathematical constant recurring in analysis and number theory, usually denoted by the Greek letter $\gamma$.

It is defined as the limiting difference between the harmonic series and the natural logarithm:

$$
\gamma=\lim _{n}\left(\sum_{k=1}^{n} \frac{1}{k}-\ln n\right)
$$

Note that in "3)" of section "6." we give a computer program that will find rational approximations to the numbers listed above.

## 6. Computer programs related to numerical sets

Now we will present some computer programs related to our subject. All these computer programs use Scilab (see [2']).

Scilab is free and open source software for numerical computation providing a powerful computing environment for engineering and scientific applications. It is available under GNU/Linux, Mac OS X and Windows XP/Vista/7. Scilab includes hundreds of mathematical functions. It has a high level programming language allowing access to advanced data structures, 2-D and 3-D graphical functions. We could deffine a script file (may contain only Scilab executable statements or both executable statements and function definitions) or a function file (function opens a function definition, endfunction closes a function definition).

1) Here is presented a way to find perfect numbers by using a script file. The idea to use the square root of $n$ is key to it's speed, because when we get to searching for perfect numbers that have magnitude of around 10000 , this will cut out $99 \%$ of the search time (i.e., we would only be searching over candidate divisors between 2 and 100 , instead of between 2 and 10000).
n_max = 10000; // Limit search range for perfect numbers
$\mathrm{n}=1$;// Initialize perfect number candidate
```
\(\mathrm{c}=0\); // Perfect number counter is set to 0
while ((c <= n_max) \& ( \(\mathrm{n}<\mathrm{n}\) _max))
        \(\mathrm{n}=\mathrm{n}+1\);
    \(\mathrm{d} 1=1 ; / / 1\) is always a divisor
    \(\mathrm{d} 2=[1] ; / / 1\) is always in the list of low divisors
    d3 = []; /The list of high divisors starts out empty
    d4 = 2 : floor(sqrt(n));
    for \(\mathrm{d}=\mathrm{d} 4\)
        if pmodulo \((\mathrm{n}, \mathrm{d})=0 / /\) Two divisors of \(n\) have been found:d and \(n / d\).
    \(\mathrm{d} 5=\mathrm{n} / \mathrm{d}\);
                \(\mathrm{d} 1=\mathrm{d} 1+\mathrm{d}+\mathrm{d} 5\);
                \(\mathrm{d} 2=[\mathrm{d} 2, \mathrm{~d}]\);
                d3 \(=[\mathrm{d} 5, \mathrm{~d} 3]\);
            end
    end
    if \(\mathrm{d} 1==\mathrm{n}\)
        \(\operatorname{disp}(\mathrm{n})\)
        \(\operatorname{disp}([\) 'It has these divisors:'])
        \(\operatorname{disp}([\mathrm{d} 2 \mathrm{~d} 3])\)
        disp(' ')
        \(\mathrm{c}=\mathrm{c}+1 ; / /\) Perfect number count increases by 1
    end
end
```

The results are shown in the following table.
Perfect numbers

| Perfect numbers $n$ | Perfect numbers |
| :---: | :---: |
| 6 | $1,2,3$ |
| 28 | $1,2,4,7,14$ |
| 496 | $1,2,4,8,16,31,62,124,248$ |
| 8128 | $1,2,4,8,16,32,64,127,254,508,1016,2032$, |

Note. $d$ is an arbitrary divisor of $n, d \neq n$, that is, $d \in \sigma(n)$.
2) In the following we will talk about the notion of computable numbers (see [5']) which are the real numbers that can be computed to within any desired precision by a finite, terminating algorithm. The computable numbers include real numbers which appear in practice, as well as $e, \pi$, and many other transcendental numbers. For example, to compute $\pi$, you could use one of the known approximation algorithms. The calculator will try to calculate $\pi$ with an accuracy of increasingly higher and this implies more digits to the screen, thing is only possible if we had an unlimited amount of memory.

But exist numbers that are non-computable. A number is non-computable if there is no program that prints its infinite decimal expansion (adding trailing zeros if a finite expansion is possible). Irrational numbers, with an infinite number of decimal places that can not be displayed by a computer program, are non-computable numbers. Question was raised as proving the existence of non-computable numbers, and the key to this thing is that there are more real numbers than computer programs. It is quite surprising considering that the set of real numbers and the set of computer programs are both infinite sets. The set computer programs that can display computable numbers is poorer than non-computable set of numbers.

The set all possible computer programs is an unusual example of a countable set. The argument for this is that every computer program is a finite sequence of symbols from a finite set of symbols. So the set $S_{n}$ of all computer programs with a length $n$ is finite. After reunion of all sets $S_{n}$ of this type we obtain the set of all computer programs $S=\bigcup_{n \geq 0} S_{n}$.

The set of all computer programs is countable, then we can assign a different integer for each program. For example, depending on length, we denote the shortest schedule 1 , next 2 , etc. If there are several programs of the same length, we sort lexicographically programs and we assign integers, in that order.

Unlike computer programs set, real numbers set is not countable. Georg Cantor was the one who showed in various ways that there is no way to assign a real number to a natural number.

There exists numbers that cannot be computed by any computer program.
3) It is sometimes desirable to approximate irrational numbers by simple rational numbers, which are fractions whose numerator and denominator are integers.

The following computer program, which uses a function type file, returns two integers $n$ the "numerator" and $d$ the "denominator", and the output argument "approx" returns the fraction $n / d$, in this way we can find "simple" rational approximations of an irrational number.

```
    function[approx,d,n]=approxfrac
(r, precision)
    \(\mathrm{a}=\) floor(r);
    \(\mathrm{r}=\mathrm{r}-\mathrm{a}\);
    \(\mathrm{pl}=0 ; \mathrm{q} 1=1\);
    \(\mathrm{p} 2=1 ; \mathrm{q} 2=1 ;\)
    \(\mathrm{b}=\mathrm{p} 1+\mathrm{p} 2\);
    \(\mathrm{d}=\mathrm{q} 1+\mathrm{q} 2\);
    while abs(r-b/d) > precision,
        if \(r>b / d\),
```

$$
\mathrm{p} 1=\mathrm{b} ; \mathrm{q} 1=\mathrm{d}
$$

else

$$
\mathrm{p} 2=\mathrm{b} ; \mathrm{q} 2=\mathrm{d}
$$

end
$\mathrm{b}=\mathrm{p} 1+\mathrm{p} 2$;
$\mathrm{d}=\mathrm{q} 1+\mathrm{q} 2$;
$\mathrm{n}=\mathrm{a}$ * $\mathrm{d}+\mathrm{b}$;
approx $=\mathrm{n} / \mathrm{d}$;
end
endfunction

Using this program we have determined integers whose fraction gives an approximation whith an error of 0.000001 of the numbers who are the subject of the open problem described earlier (see 5.2.). The results are shown in the following table.

Irational numbers

| Numbers | Numerator | Denominator | Approximate fraction numbers |
| :---: | :---: | :---: | :---: |
| $\pi$ | 355 | 113 | 3.141592 |
| $\pi \mathrm{e}$ | 2579 | 302 | 8.539734 |
| $\pi / \mathrm{e}$ | 7325 | 6338 | 1.155727 |
| $\rho^{\wedge}$ | 13221 | 2009 | 6.580886 |
| $\pi^{\wedge}$ | 44267 | 1971 | 22.459157 |
| $\pi^{\sqrt{2}}$ | 13815 | 2737 | 5.047497 |
| $\pi^{-}$ | 36124 | 991 | 36.462159 |
| $e^{\pi^{\imath}}$ | 72191995 | 3734 | 19333.689073 |
| $e^{e}$ | 11199 | 739 | 15.154263 |
| $\ln \pi$ | 1922 | 1679 | 1.144729 |
| $C$ | 228 | 395 | 0.5772152 |

4) In the following we highlight that the set $\mathbb{Q}$ of rational numbers is countable. This property of $\mathbb{Q}$ allows us to say that although the set $\mathbb{Q}$ is dense in $\mathbb{R}$ see P1. in 2.3. - however it is quite sparse according, for example, to [6']. Indeed the set of real numbers and the set of irrational numbers are not countable. There is no way to assign a different integer to each real number or irrational number. Below is an intuitive diagram that should tell us how to do this counting for the set of rational numbers, the method known under the name of Cantor's enumeration of a countable collection of countable sets.

$$
\left.\begin{array}{llll}
\frac{1}{1} \rightarrow \frac{1}{2} & \frac{1}{3} \rightarrow & \frac{1}{4} \ldots \frac{1}{n} \ldots \\
\swarrow & \nearrow & \swarrow \\
\frac{2}{1} & \left(\frac{2}{2}\right. & \frac{2}{3} & \frac{2}{4} \ldots \frac{2}{n} \\
\downarrow & \nearrow \\
\frac{3}{1} & \frac{3}{2} & \frac{3}{3} & \frac{3}{4} \ldots
\end{array}\right] \frac{3}{n} \ldots
$$

$$
\downarrow \nearrow \swarrow \quad(\alpha)
$$

We remark that the rational $\frac{a}{b}$ numbers in $(\alpha)$ related by arrows and situated on the same "diagonal" are exactly those such that $a+b=p$ with $p=3,4,5 \ldots$

Also, we will identify the fraction $\frac{a}{b}$ with the couple $(a, b)$ (ordered pair). In this way we have put every point $(a, b)$ in one-to-one coorespondence with the set of natural numbers. The diagram below illustrates this process of counting.

$$
\begin{array}{llll}
(4,1) & (4,2) & (4,3) & (4,4) \\
(3,1) & (3,2) & (3,3) & (3,4) \\
(2,1) & (2,2) & (2,3) & (2,4) \\
(1,1) & (1,2) & (1,3) & (1,4)
\end{array}
$$

We will give a computer program that counts a number of fractions selected by us. Fractions having not numerators and denominators as coprime numbers between them will be identified using Euclid's algorithm and will be eliminated from the count. (Recall that two integers $n, d$ are said to be coprime, or relative prime, if the only positive integer that divides both of them is 1. Also, we mention that Euclid's algorithm, or Euclidean algorithm, is an efficient method for computing the greatest common divisor of two integers. It is named after the Greek mathematician Euclid $\sim 300 \mathrm{BC}$. This algorithm starts with a pair of positive integers and forms a new pair that consists of the smaller number and the difference between the larger and smaller numbers. The process repeats until the obtained numbers are equal. The number then is the greatest common divisor of the original pair.)

```
function \([\mathrm{x}]=\underline{\mathrm{frac}}(\mathrm{n}, \mathrm{d})\)
        \(\mathrm{x}=(\mathrm{n} * \% \mathrm{~s}) /\left(\mathrm{d}^{*} \% \mathrm{~s}\right) ;\)
endfunction
function [n1]=euclid(n1, n2)
    // n1 and n2 are positive integers
    if \(\mathrm{n} 2>\mathrm{n} 1\)
        tem \(=\mathrm{n} 2 ; \mathrm{n} 2=\mathrm{n} 1 ; \mathrm{n} 1=\) tem; // to ensure \(\mathrm{n} 1>=\mathrm{n} 2\)
    end
    \(\mathrm{r}=\) pmodulo \((\mathrm{n} 1, \mathrm{n} 2)\); // remainder when n 2 divides n 1
    \(\mathrm{n} 1=\mathrm{n} 2 ; \mathrm{n} 2=\mathrm{r}\);
    while \(\mathrm{r} \sim=0\)
        \(\mathrm{r}=\) pmodulo \((\mathrm{n} 1, \mathrm{n} 2)\);
        \(\mathrm{n} 1=\mathrm{n} 2 ; \mathrm{n} 2=\mathrm{r}\);
    end
endfunction
write(\%io(2),");
write(\%io(2),'Define the set of fractions.');
write(\%io(2),'The maximum value for the numerator: ','(a)');
```

```
a=read(%io(1),1,1);
write(%io(2),' The maximum value for the denominator: ','(a)');
b=read(%io(1),1,1);
write(%io(2),");
write(%io(2),'Define the fraction of the set.');
write(%io(2),'Numerator: ','(a)');
count_a=read(%io(1),1,1);
write(%io(2),'Denominator: ','(a)');
count_b=read(%io(1),1,1);
write(%io(2),");
count=0
found=0
if (count_a~=count_b) then
    y=euclid(count_a,count_b);
    if (y~=1) then
        count_a=count_a/y;
        count_b=count_b/y;
    end
end
for i=1:a
    for j=1:b
        x= euclid(i,j);
        if (i==j)|(x==1) then
// current_frac = frac(i,j);
// disp(current_frac);
            count=count+1;
            if (i==count_a)&(j==count_b) then
                    write(%io(2),'Fraction has index: ','(a)');
                    write(%io(2),count,'(fl0.0)');
                    found=1
            end
        end
    end
end
if (found==0) then
    write(%io(2),'Fraction does not belong to the set defined above. ','(a)');
end
```

Applying this program for fractions having numerators and denominators smaller or equal to 10 were returned the following results, which appear in the table below. Near to each fraction is writing its ordering number.

Indexing fractions

| $\frac{1}{1} \rightarrow{ }_{1}$ | $\frac{1}{2} \rightarrow 2$ | $\frac{1}{3} \rightarrow 3$ | $\frac{1}{4} \rightarrow 4$ | $\frac{1}{5} \rightarrow 5$ | $\frac{1}{6} \rightarrow 6$ | $\frac{1}{7} \rightarrow 7$ | $\frac{1}{8} \rightarrow 8$ | $\frac{1}{9} \rightarrow 9$ | $\frac{1}{10} \rightarrow 1$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 |  |  |  |  |  |  |  |  |  |
| $\frac{2}{1} \rightarrow 11$ | $\frac{2}{2} \rightarrow 1$ | $\frac{2}{3} \rightarrow 13$ | $\frac{2}{4} \rightarrow 2$ | $\frac{2}{5} \rightarrow 14$ | $\frac{2}{6} \rightarrow 3$ | $\frac{2}{7} \rightarrow 15$ | $\frac{2}{8} \rightarrow 4$ | $\frac{2}{9} \rightarrow 16$ | $\frac{2}{10} \rightarrow 5$ |
| $\frac{3}{1} \rightarrow 17$ | $\frac{3}{2} \rightarrow 18$ | $\frac{3}{3} \rightarrow 1$ | $\frac{3}{4} \rightarrow 20$ | $\frac{3}{5} \rightarrow 21$ | $\frac{3}{6} \rightarrow 2$ | $\frac{3}{7} \rightarrow 22$ | $\frac{3}{8} \rightarrow 23$ | $\frac{3}{9} \rightarrow 3$ | $\frac{3}{10} \rightarrow 24$ |
| $\frac{4}{1} \rightarrow 25$ | $\frac{4}{2} \rightarrow 11$ | $\frac{4}{3} \rightarrow 26$ | $\frac{4}{4} \rightarrow 1$ | $\frac{4}{5} \rightarrow 28$ | $\frac{4}{6} \rightarrow 13$ | $\frac{4}{7} \rightarrow 29$ | $\frac{4}{8} \rightarrow 2$ | $\frac{4}{9} \rightarrow 30$ | $\frac{4}{10} \rightarrow 14$ |
| $\frac{5}{1} \rightarrow 31$ | $\frac{5}{2} \rightarrow 32$ | $\frac{5}{3} \rightarrow 33$ | $\frac{5}{4} \rightarrow 34$ | $\frac{5}{5} \rightarrow 1$ | $\frac{5}{6} \rightarrow 36$ | $\frac{5}{7} \rightarrow 37$ | $\frac{5}{8} \rightarrow 38$ | $\frac{5}{9} \rightarrow 39$ | $\frac{5}{10} \rightarrow 2$ |
| $\frac{6}{1} \rightarrow 40$ | $\frac{6}{2} \rightarrow 17$ | $\frac{6}{3} \rightarrow 11$ | $\frac{6}{4} \rightarrow 18$ | $\frac{6}{5} \rightarrow 41$ | $\frac{6}{6} \rightarrow 1$ | $\frac{6}{7} \rightarrow 43$ | $\frac{6}{8} \rightarrow 20$ | $\frac{6}{9} \rightarrow 13$ | $\frac{6}{10} \rightarrow 21$ |
| $\frac{7}{1} \rightarrow 44$ | $\frac{7}{2} \rightarrow 45$ | $\frac{7}{3} \rightarrow 46$ | $\frac{7}{4} \rightarrow 47$ | $\frac{7}{5} \rightarrow 48$ | $\frac{7}{6} \rightarrow 49$ | $\frac{7}{7} \rightarrow 1$ | $\frac{7}{8} \rightarrow 51$ | $\frac{7}{9} \rightarrow 52$ | $\frac{7}{10} \rightarrow 53$ |
| $\frac{8}{1} \rightarrow 54$ | $\frac{8}{2} \rightarrow 25$ | $\frac{8}{3} \rightarrow 55$ | $\frac{8}{4} \rightarrow 11$ | $\frac{8}{5} \rightarrow 56$ | $\frac{8}{6} \rightarrow 26$ | $\frac{8}{7} \rightarrow 57$ | $\frac{8}{8} \rightarrow 1$ | $\frac{8}{9} \rightarrow 59$ | $\frac{8}{10} \rightarrow 28$ |
| $\frac{9}{1} \rightarrow 60$ | $\frac{9}{2} \rightarrow 61$ | $\frac{9}{3} \rightarrow 17$ | $\frac{9}{4} \rightarrow 62$ | $\frac{9}{5} \rightarrow 63$ | $\frac{9}{6} \rightarrow 18$ | $\frac{9}{7} \rightarrow 64$ | $\frac{9}{8} \rightarrow 65$ | $\frac{9}{9} \rightarrow 1$ | $\frac{9}{10} \rightarrow 67$ |
| $\frac{10}{1} \rightarrow 68$ | $\frac{10}{2} \rightarrow 31$ | $\frac{10}{3} \rightarrow 69$ | $\frac{10}{4} \rightarrow 32$ | $\frac{10}{5} \rightarrow 11$ | $\frac{10}{6} \rightarrow 33$ | $\frac{10}{7} \rightarrow 70$ | $\frac{10}{8} \rightarrow 3$ | $\frac{10}{9} \rightarrow 71$ | $\frac{10}{10} \rightarrow 1$ |

## 7. An application in engineering with a computer program

The main application of the concept of a countable set in engineering is in electrical engineering, namely the conversion from analog to digital (A/D conversion) of a signal. In fact, this conversion is the reduction of a continuous signal to a discrete signal, what is known as sampling.

Actually, sampling appears in signal processing, which is an area of systems engineering, electrical engineering and applied mathematics that deals with operations on or analysis of signals, or measurements of time-varying or spatially-varying physical quantities. (Signals of interest can include sound, electromagnetic radiation,
images, and sensor data, for example biological data such as electrocardiograms, control system signals, telecommunication transmission signals, and many others.) For example, it is known that digital signal is required for storage and/or transmission of a signal.

A common example of sampling is the conversion of a sound wave (a continuous signal) to a set of samples (a discrete-time signal). A sample refers to a value or set of values at a point in time and/or space. A sampler is a subsystem or operation that extracts samples from a continuous signal.

A theoretical ideal sampler produces samples equivalent to the instantaneous values of the continuous signal at the desired points.

To be more clearly, in electrical engineering, a (continuous) function like $f(t)=\cos t$, where $t$ is the time in seconds, is referred to as an analog signal. To digitize the signal, we sample $f(t)$ every $\wedge t$ seconds ( $\wedge t$ being called the sampling interval), to form a countable set:

$$
D=\{f(\Delta t), f(2 \Delta t), f(3 \Delta t), \ldots, f(n \Delta t), \ldots\} .
$$

(It is obviously that the set $D$ is countable, because we can easily determined an one-to-one correspondence between $\mathbb{N}$ and $D$, namely the function $g$ which maps each nonzero natural number $n$ in $f(n \Delta t)$ ).

For example, sampling $f(t)=\cos t$, of every $\Lambda t=0.1$ seconds produces the set:

$$
\left\{\cos \frac{1}{10} \cdot \cos \frac{2}{10} \cdot \cos \frac{3}{10} \cdots \cdot \cos \frac{n}{10} \cdots\right\} .
$$

## A computer program for an ideal sampler

In the following examples we will give the first 38 terms of a sampling of a signal every $\Delta t$ seconds and the graphs of respective functions represented by using the values obtained by sampling.

We can write a mathematical expression for the voltage signal as a function of time denoted for example by $f(t)$. If the signal is sinusoidal, a general form of the $f(t)$ is the following

$$
f(t)=A \sin (\omega t+\varphi) .
$$

The signal has three parameters: the amplitude $A$, the angular frequency $\omega$ and the phase angle $\varphi$.

Amplitude (in volts) of a sinusoidal signal is the largest value it takes (the sine function has the values in the interval $[-1,1]$ so that $f(t)$ takes its values in the interval $[-|A|,|A|]$.

Angular frequency (in hertz or cycles/s) is a parameter that determines how often the sinusoidal signal goes through a cycle.

Phase angle (in radians) is the initial angle of a sinusoidal function at its origin.
The following four images represent in the "usual manner" some first terms of a sampling of the signals respectively modeled by the following functions:
I) a) $f(t)=\sin t, \Delta t=0.5(A=1 . \omega=1 .(\rho=0) ; \mathbf{b}) f(t)=3 \sin t, \Delta t=0.5(A=3 . \omega=1 .(\rho=0)$; c) $f(t)=\sin 3 t, \Delta t=0.5 \quad(A=1 . \omega=3 . \rho=0) ;$ d) $f(t)=\sin \frac{t}{3}, \Delta t=0.5\left(A=1, \omega=\frac{1}{3}, \varphi=0\right)$.

The usual manner to plot the sampling of these signals is by vertical bars, corresponding of all magnitudes of $f$ in the points $n \cdot \Delta t$. (The maximum of these magnitudes is the amplitude of this signal.)


$3 \sin (t)$
$\sin (3 t)$

$\sin \left(\frac{t}{3}\right)$

Graphical Representation of Sinusoidal Signals

```
    t=0:0.5:6*%pi;
    x=}\operatorname{sin}(\textrm{t});\textrm{y}=\operatorname{sin}(3*\textrm{t});\textrm{z}=3*\operatorname{sin}(\textrm{t});\textrm{u
=sin(t/3)
    subplot(2,2,1)
    a=gca()
    a.thickness=2;
    a.x_location = "origin";
    a.y_location = "origin";
    a.data_bounds=[0,-3;20,3]
    plot2d3('gnn',t,x)
    subplot(2,2,2)
    a=gca()
    a.thickness=2;
    a.x_location = "origin";
    a.y_location = "origin";
    a.data_bounds=[0,-3;20,3]
```

```
plot2d3('gnn',t,z)
subplot(2,2,3)
a=gca()
a.thickness=2;
a.x_location = "origin";
a.y_location = "origin";
a.data_bounds=[0,-3;20,3]
plot2d3('gnn',t,y)
subplot(2,2,4)
a=gca()
a.thickness=2;
a.x_location = "origin";
a.y_location = "origin";
a.data_bounds=[0,-3;20,3]
plot2d3('gnn',t,u)
```

Now we consider the first sampling for $f(t)=\sin t$ and figure the points at extremities of vertical bars, extremities which are not on the horizontal axis. These points that appear in this illustration form a subset of a countable set.


Graphical Representation of $\sin t$
Next, we retain only points figured in the last illustration and we figure curve that is found, that is the graph of the function sint. Do the same with other signals that is $3 \sin t, \sin 3 t, \sin \left(\frac{t}{3}\right)$, respectively.


In the following image are represented all four original sinusoidal signals not sampled. It is easy to see that are the amplitudes and the angular frequencies of these signals. Remark that all these signals are periodic. Their fundamental periods are related to their angular frequencies; indeed, is well known that the fundamental period
of the sinusoid $f(t)=A \sin (\omega t+\infty)$ is $\frac{2 \pi}{\omega}$. So the fundamental periods for $\sin t, 3 \sin t, \sin 3 t, \sin \frac{t}{3}$ are $2 \pi, 2 \pi, \frac{2 \pi}{3}$ and $\frac{2 \pi}{1}=6 \pi$ respectively.

II)
a) $f(t)=\frac{\sin t}{t}, \Delta t=0.5$; b) $f(t)=\frac{t g t}{t}, \Delta t=0.5$;

If the sinusoidal signals above are periodic signals, that is they repeat in time, in the sequel we have some not periodic signals (sinusoidal signals aren't always the most interesting kind of signals). The next two signals are modeled as $\frac{\sin t}{t}$ and $\frac{\operatorname{tg} t}{t}$. The points marked on these graphs are sampling of the corresponding signals. In fact for each signal these points form a subset of a countable set. Remark that $\lim _{t \rightarrow 0} \frac{\sin t}{t}=1=\lim _{t \rightarrow 0} \frac{\operatorname{tg} t}{t}$ (known in mathematical analysis as "fundamental limits").

III) a) $f(t)=t^{2}, \Delta t=0.5$; b) $f(t)=(t+2)^{2}, \Delta t=0.5$; c) $f(t)=(t-2)^{2}, \Delta t=0.5$;
d) $f(t)=t^{2}+2, \Delta t=0.5$; e) $f(t)=t^{2}-2, \Delta t=0.5$.

The following image is for five second-order polinomial signals. These functions are represented in the same reference axes. Note that the graphs of these signals (in their original forms, that is not sampled) can be obtained by translating the graph of the function $f(t)=t^{\prime}$. Indeed, we can obtain $(t+2)^{\prime} .(t-2)^{\prime} . t^{2}+2$. $t^{2}-2$ by translating $f(t)$ with two units to the left, to the right, up and down, respectively.

## 8. Conclusion

This paper proves that teaching mathematics in a technical university can sometimes be quite difficult. This is because the classical exposure, just technique is not enough. Without sacrificing rigor, we can increase availability and attractiveness of exposure adding some informal parties. And is not easy, for example, that notions that appear in these parties have themselves an informal description as to be understood by persons who are not specialists in the field of pure or applied mathematics.

But the biggest challenge of teaching mathematics for engineers is that of finding a common language between mathematics and engineering disciplines. Therefore, step by step, persons who teach mathematics should indentify punctual applications in technical disciplines. Our paper illustrates how we can do this for one of concepts presented, namely the notion of countable set.

Some graphical representations accompanied by a computer program performs a suggestive application for what means a "countable set", namely samplings of some (not necessarily periodic) signals. We can also "see" that two of the fundamental limits of the elementary mathematical analysis, namely $\lim _{x \rightarrow 0} \frac{\sin x}{x}$ and $\lim _{x \rightarrow 0} \frac{\operatorname{tg} x}{x}$ have value 1 . The subject also gives us the opportunity to suggest what the geometric transformation called translation can be applied to generate a class of graphics from a fundamental one. Three main reasons led us to the need to consider the approximation of some well-known real (irrational) numbers by rationals:

1) Real numbers underlying mathematics.
2) We can not talk today about engineering design without the use of computers
3) Irrational numbers are represented approximately in a computer and running programs uses sometimes these approximations (although some irrationals can be manipulated to obtain often "exact" results).

And, because the set of rational numbers (who serve as material of these approximations) is a beautiful example of a countable set, a computer program transcribes the most common proof of this. (This proof is based on Cantor's enumeration of a countable collection of countable sets.)

As a final conclusion, the construction of our work follows some connected ideas: using mathematics and computer science in engineering, sets of real numbers and the representation of reals in a computer, the approximation of irrational numbers by rationals, the countability of the set of rationals and of the set of computer programs; and, in consequence, our choice to find an application of the concept of countable set, in engineering.

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